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The Many Faces of Maxwell, Dirac and Einstein Equations

A Clifford Bundle Approach

Second Edition

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