

Eigenvalues, Multiplicities and Graphs

CHARLES R. JOHNSON

College of William and Mary, Williamsburg, Virginia

CARLOS M. SAIAGO

Universidade Nova de Lisboa



CAMBRIDGE
UNIVERSITY PRESS

Contents

<i>Preface</i>	<i>page</i>	xiii
<i>List of Terms and Symbols</i>		xvii
0	Background	1
0.1	Matrices	1
0.1.1	Hermitian / Real Symmetric Matrices	2
0.1.2	Interlacing Eigenvalues	2
0.1.3	Rank Inequalities and Change in Hermitian Multiplicities	3
0.1.4	Eigenvector Structure When a Submatrix Has the Same Eigenvalue	3
0.1.5	Perron-Frobenius Theory of Nonnegative Matrices	4
0.1.6	Entries of Matrix Powers	4
0.1.7	M-matrices	4
0.2	Graphs	5
0.2.1	Definitions	5
0.2.2	Trees	6
0.2.3	Graphs and Matrices	7
0.2.4	Graphs and Characteristic Polynomial Formulae	8
0.3	Other Background	8
1	Introduction	10
1.1	Problem Definition	10
1.2	Matrices versus Graphs	11
1.3	Early History	12
1.4	The Interlacing Constraint	13
1.5	Overview	14

2	Parter-Wiener, etc. Theory	16
2.1	Introduction	16
2.2	An Example	16
2.3	General Theory of the Existence of Parter Vertices for Trees	18
2.4	Characterization of Parter Vertices	25
2.5	The Possible Changes in Status of One Vertex upon Removal of Another	28
2.6	At Least Two Multiplicities Equal to 1	42
2.7	Eigenstructure of Tridiagonal Hermitian Matrices and Their Principal Submatrices	45
2.8	Nontrees	47
2.9	Tree-Like Vertices	49
3	Maximum Multiplicity for Trees, I	51
3.1	Introduction	51
3.2	Path Covers and Path Trees	51
3.3	$\Delta(T) = \text{Maximum } p - q$	53
3.4	$M(T) = P(T), \Delta(T), n - \text{mr}(T)$	58
3.5	Calculation of $M(T)$ and Bounds	60
3.5.1	Calculation of $M(T)$ in Linear Time	61
3.5.2	Estimation of $M(T)$ from the Degree Sequence of T	64
4	Multiple Eigenvalues and Structure	69
4.1	Perturbation of Diagonal Entries and Vertex Status	69
4.2	Parter Vertices, Parter Sets and Fragmentation	74
4.3	The Fundamental Decomposition	79
4.4	Eigenspace Structure and Vertex Classification	82
4.5	Removal of an Edge	90
4.5.1	Basic Inequalities	90
4.5.2	Classification of Edges in Trees Based on the Classification of Their Vertices	95
5	Maximum Multiplicity, II	96
5.1	The Structure of Matrices with a Maximum Multiplicity Eigenvalue	96
5.2	NIM Trees	101
5.3	The Second Maximum Multiplicity	108
6	The Minimum Number of Distinct Eigenvalues	110
6.1	Introduction	110
6.2	The Diameter and a Lower Bound for $c(T)$	110

6.3	The Method of Branch Duplication: Combinatorial and Algebraic	112
6.4	Converse to the Diameter Lower Bound for Trees	122
6.5	Trees of Diameter 7	127
6.6	The Function $C(d)$ and Disparity	129
6.7	The Minimum Number of Multiplicities Equal to 1	132
6.8	The Relative Position of Multiple Eigenvalues in Ordered Lists	134
6.8.1	A Lower Bound for the Cardinality of a Fragmenting Parter Set	134
6.8.2	The Relative Position of a Single Multiple Eigenvalue	136
6.8.3	Vertex Degrees	140
6.8.4	Two Multiple Eigenvalues	144
7	Construction Techniques	146
7.1	Introduction	146
7.2	Eigenvalues for Paths and Subpaths	146
7.3	The Method of Assignments	147
7.4	Derivation of a Multiplicity List via Assignment: An Example	149
7.5	A 13-Vertex Example	150
7.6	The Implicit Function Theorem (IFT) Approach	151
7.7	More IFT, Examples, Vines	156
7.8	Polynomial Constructions	160
8	Multiplicity Lists for Generalized Stars	167
8.1	Introduction	167
8.2	A Characterization of Generalized Stars	168
8.3	The Case of Simple Stars	169
8.4	An Inverse Eigenvalue Problem for Generalized Stars	173
8.5	The Multiplicity Lists	174
8.6	The IEP versus Ordered Multiplicity Lists	177
8.7	The Upward Multiplicity Lists	181
8.8	$c(T)$ and $U(T)$	183
9	Double Generalized Stars	186
9.1	Introduction	186
9.2	Observations about Double Generalized Stars	187
9.3	The Multiplicity Lists	190
9.4	Double Paths	196

10	Linear Trees	200
	10.1 Introduction	200
	10.2 The Second Superposition Principle for Linear Trees	201
	10.3 Possible Multiplicity Lists for Linear Trees	203
	10.4 Cases of Sufficiency of Linear Trees	207
	10.5 Special Results for Linear Trees	209
11	Nontrees	211
	11.1 Introduction and Observations	211
	11.2 The Complete Graph	211
	11.3 The Cycle	213
	11.4 A Tree + an Edge	214
	11.4.1 A Graph + an Edge	220
	11.5 The Graphs G for Which $M(G) = 2$	222
	11.6 Graphs Permitting Just Two Distinct Eigenvalues	225
	11.7 Nearly Complete Graphs	228
12	Geometric Multiplicities for General Matrices over a Field	232
	12.1 Preliminaries	232
	12.2 Geometric Parter-Wiener, etc. Theory	234
	12.3 The Geometric Downer Branch Mechanism for General Matrices over a Field	239
	12.4 The Maximum Geometric Multiplicity for a Tree	243
	12.5 The Minimum Number of Distinct Eigenvalues in a Diagonalizable Matrix Whose Graph Is a Tree	245
	Appendix A: Multiplicity Lists for Trees on Fewer Than 12 Vertices	247
	A.1 Tree on 3 Vertices (1 tree)	247
	A.2 Trees on 4 Vertices (2 trees)	247
	A.3 Trees on 5 Vertices (3 trees)	247
	A.4 Trees on 6 Vertices (6 trees)	248
	A.5 Trees on 7 Vertices (11 trees)	248
	A.6 Trees on 8 Vertices (23 trees)	248
	A.7 Trees on 9 Vertices (47 trees)	250
	A.8 Trees on 10 Vertices (106 trees)	253
	A.9 Trees on 11 Vertices (235 trees)	259
	Appendix B: Seeds for Branch Duplication	276
	B.1 Diameter < 7 Seeds	276
	B.2 Diameter 7 Seeds and Classification of Their Families Using Assignments	277

B.3	Unfoldings in Each of the Three Families for Which $c(T)$ Is Demonstrably 8	279
	<i>Bibliography</i>	281
	<i>Index</i>	287