

**Mathematical  
Surveys  
and  
Monographs**  
Volume 219

# The Projective Heat Map

**Richard Evan Schwartz**



**American Mathematical Society**  
Providence, Rhode Island

# Contents

Preface	ix
Chapter 1. Introduction	1
1.1. From Geometry to Dynamics	1
1.2. The Projective Heat Map	3
1.3. A Picture of the Julia Set	4
1.4. The Core Results	4
1.5. Deeper Structure	6
1.6. A Few Corollaries	8
1.7. Sketch of the Proofs	8
1.8. Some Comparisons	9
1.9. Outline of the Monograph	10
1.10. Companion Program	11
<b>Part 1.</b>	<b>13</b>
Chapter 2. Some Other Polygon Iterations	15
2.1. The Midpoint Theorem	15
2.2. The Midpoint Iteration	15
2.3. Napoleon's Theorem	17
2.4. Napoleon's Iteration	18
2.5. Conformal Averaging	19
Chapter 3. A Primer on Projective Geometry	23
3.1. The Real Projective Plane	23
3.2. Affine Patches	23
3.3. Projective Transformations and Dualities	24
3.4. The Cross Ratio	24
3.5. The Hilbert Metric	25
3.6. Projective Invariants of Polygons	27
3.7. Duality and Relabeling	29
3.8. The Gauss Group	30
Chapter 4. Elementary Algebraic Geometry	31
4.1. Measure Zero Sets	31
4.2. Rational Maps	31
4.3. Homogeneous Polynomials	32
4.4. Bezout's Theorem	32
4.5. The Blow-up Construction	33

Chapter 5. The Pentagram Map	37
5.1. The Pentagram Configuration Theorem	37
5.2. The Pentagram Map in Coordinates	37
5.3. The First Pentagram Invariant	39
5.4. The Poincare Recurrence Theorem	40
5.5. Recurrence of the Pentagram Map	41
5.6. Twisted Polygons	42
5.7. The Pentagram Invariants	42
5.8. Symplectic Manifolds and Torus Motion	43
5.9. Complete Integrability	44
 Chapter 6. Some Related Dynamical Systems	 47
6.1. Julia Sets of Rational Maps	47
6.2. The One-Sided Shift	48
6.3. The Two-Sided Shift	51
6.4. The Smale Horseshoe	51
6.5. Quasi Horseshoe Maps	52
6.6. The 2-adic Solenoid	58
6.7. The BJK Continuum	59
 <b>Part 2.</b>	 61
 Chapter 7. The Projective Heat Map	 63
7.1. The Reconstruction Formula	63
7.2. The Dual Map	64
7.3. Formulas for the Projective Heat Map	65
7.4. The Case of Pentagons	67
7.5. Some Speculation	68
 Chapter 8. Topological Degree of the Map	 71
8.1. Overview	71
8.2. The Lower Bound	71
8.3. The Upper Bound	72
 Chapter 9. The Convex Case	 75
9.1. Flag Invariants of Convex Pentagons	75
9.2. The Gauss Group Acting on the Unit Square	76
9.3. A Positivity Criterion	76
9.4. The End of the Proof	78
9.5. The Action on the Boundary	80
9.6. Discussion	80
 Chapter 10. The Basic Domains	 81
10.1. The Space of Pentagons	81
10.2. The Action of the Gauss Group	82
10.3. Changing Coordinates	83
10.4. Convex and Star Convex Classes	84
10.5. The Semigroup	84
10.6. A Global Point of View	86

Chapter 11. The Method of Positive Dominance	89
11.1. The Divide and Conquer Algorithm	89
11.2. Positivity	91
11.3. The Denominator Test	91
11.4. The Area Test	93
11.5. The Expansion Test	93
11.6. The Confinement Test	94
11.7. The Exclusion Test	95
11.8. The Cone Test	95
11.9. The Stretch Test	96
Chapter 12. The Cantor Set	97
12.1. Overview	97
12.2. The Big Disk	98
12.3. The Six Small Disks	99
12.4. The Diffeomorphism Property	101
12.5. The Main Argument	104
12.6. Proof of the Measure Expansion Lemma	105
12.7. Proof of the Metric Expansion Lemma	105
12.8. Discussion	107
Chapter 13. Towards the Quasi Horseshoe	109
13.1. The Target	109
13.2. The Outer Layer	109
13.3. The Inner Layer	111
13.4. The Last Three pieces	113
Chapter 14. The Quasi Horseshoe	115
14.1. Overview	115
14.2. Existence of The Quasi Horseshoe	115
14.3. The Invariant Cantor Band	117
14.4. Covering Property	118
14.5. Subspace Property	118
14.6. Attracting Property	119
<b>Part 3.</b>	<b>121</b>
Chapter 15. Sketches for the Remaining Results	123
15.1. The General Setup	123
15.2. The Solenoid Result	124
15.3. Local Structure	126
15.4. The Embedded Graph	127
15.5. Path Connectivity	128
15.6. The Postcritical Set	128
15.7. No Rational Fibration	129
Chapter 16. Towards the Solenoid	131
16.1. The Four Strips	131
16.2. Two Cantor Cones	132
16.3. Using Symmetry	135

16.4.	The Limiting Arc	138
Chapter 17.	The Solenoid	141
17.1.	Recognizing the BJK Continuum	141
17.2.	Taking Covers	142
17.3.	Connectivity and Unboundedness	143
17.4.	The Canonical Loop	144
17.5.	Using Symmetry for the Cone Points	144
17.6.	The First Cone Point	145
17.7.	The Second Cone Point	146
Chapter 18.	Local Structure of the Julia Set	149
18.1.	Blowing Down the Exceptional Fibers	149
18.2.	Everything but One Piece	151
18.3.	The Last Piece	151
18.4.	The Last Point	157
18.5.	Some Definedness Results	160
Chapter 19.	The Embedded Graph	161
19.1.	Defining the Generator	161
19.2.	From Generator to Edge	166
19.3.	From Edge to Pentagon	167
19.4.	Pre-images of the Pentagon	168
19.5.	The First Connector	169
19.6.	The Second Connection	170
19.7.	The Third Connector	172
19.8.	The End of the Proof	173
Chapter 20.	Connectedness of the Julia Set	175
20.1.	The Region Between the Disks	175
20.2.	The Local Diffeomorphism Lemma	179
20.3.	A Case by Case Analysis	181
20.4.	The Final Picture	185
Chapter 21.	Terms, Formulas, and Coordinate Listings	187
21.1.	Symbols and Terms	187
21.2.	Two Important Numbers	189
21.3.	The Maps	189
21.4.	Some Special Points	189
21.5.	The Cantor Set Pieces	190
21.6.	The Horseshoe Pieces	190
21.7.	The Refinement	192
21.8.	Auxiliary Polygons	192
References		193