Contents

Acknowledgments vii

Preface ix

Overview 1

1 What is a group? 3

1.1 A famous toy . . . 3
1.2 Considering the cube . . . 4
1.3 The study of symmetry . . . 5
1.4 Rules of a group . . . 6
1.5 Exercises . . . 7

2 What do groups look like? 11

2.1 Mapmaking . . . 11
2.2 A not-so-famous toy . . . 14
2.3 Mapping a group . . . 15
2.4 Cayley diagrams . . . 18
2.5 A touch more abstract . . . 19
2.6 Exercises . . . 21

3 Why study groups? 25

3.1 Groups of symmetries . . . 25
3.2 Groups of actions . . . 34
3.3 Groups everywhere . . . 36
3.4 Exercises . . . 37

4 Algebra at last 41

4.1 Where have all the actions gone? . . . 41
4.2 Combine, combine, combine . . . 44
4.3 Multiplication tables . . . 45
4.4 The classic definition . . . 48
4.5 Exercises . . . 52

5 Five families **63**
5.1 Cyclic groups . . . 64
5.2 Abelian groups . . . 68
5.3 Dihedral groups . . . 74
5.4 Symmetric and alternating groups . . . 78
5.5 Exercises . . . 87

6 Subgroups **97**
6.1 What multiplication tables say about Cayley diagrams . . . 97
6.2 Seeing subgroups . . . 99
6.3 Revealing subgroups . . . 101
6.4 Cosets . . . 102
6.5 Lagrange's theorem . . . 105
6.6 Exercises . . . 108

7 Products and quotients **117**
7.1 The direct product . . . 117
7.2 Semidirect products . . . 128
7.3 Normal subgroups and quotients . . . 132
7.4 Normalizers . . . 140
7.5 Conjugacy . . . 142
7.6 Exercises . . . 147

8 The power of homomorphisms **157**
8.1 Embeddings and quotient maps . . . 157
8.2 The Fundamental Homomorphism Theorem . . . 167
8.3 Modular arithmetic . . . 169
8.4 Direct products and relatively prime numbers . . . 172
8.5 The Fundamental Theorem of Abelian Groups . . . 175
8.6 Semidirect products revisited . . . 177
8.7 Exercises . . . 179

9 Sylow theory **193**
9.1 Group actions . . . 194
9.2 Approaching Sylow: Cauchy's Theorem . . . 199
9.3 p-groups . . . 205
9.4 Sylow Theorems . . . 208
9.5 Exercises . . . 217

10 Galois theory **221**
10.1 The big question . . . 221
10.2 More big questions . . . 225
10.3 Visualizing field extensions . . . 228
10.4 Irreducible polynomials . . . 231
10.5 Galois groups . . . 233
10.6 The heart of Galois theory . . . 243
10.7 Unsolvability . . . 247
10.8 Exercises . . . 252

A Answers to selected Exercises **261**

Bibliography **285**

Index of Symbols Used **287**

Index **289**

About the Author **297**