Contents

1 The Simplest Examples 1
1.1 Symbol Spaces and Bernoulli Shifts 5

2 Dynamical Properties of Measurable Transformations 9
2.1 The Basic Definitions 9
2.2 Recurrent, Conservative, and Dissipative Systems 16
2.2.1 Ergodicity 18
2.2.2 Kac's Lemma 19
2.2.3 Conservativity and Hopf Decomposition 21
2.3 Noninvertible Maps and Exactness 23

3 Attractors in Dynamical Systems 27
3.1 Attractors 27
3.2 Examples of Attractors 30
3.3 Sensitive Dependence, Chaotic Dynamics, and Turbulence 33
3.3.1 Unimodal Interval Maps 36

4 Ergodic Theorems 41
4.1 The Koopman Operator for a Dynamical System 41
4.2 Von Neumann Ergodic Theorems 43
4.3 Birkhoff Ergodic Theorem 47
4.4 Spectrum of an Ergodic Dynamical System 51
4.5 Unique Ergodicity 53
4.5.1 The Topology of Probability Measures on Compact Metric Spaces 54
4.6 Normal Numbers and Benford's Law 58
4.6.1 Normal Numbers 59
4.6.2 Benford's Law 60
4.6.3 Detecting Financial Fraud Using Benford's Law 62

5 Mixing Properties of Dynamical Systems 65
5.1 Weak Mixing and Mixing 67
5.2 Noninvertibility 76
5.2.1 Partitions 77
5.2.2 Rohlin Partitions and Factors 79
5.3 The Parry Jacobian and Radon–Nikodym Derivatives 81
5.4 Examples of Noninvertible Maps 84
5.5 Exact Endomorphisms 85

6 Shift Spaces 89
6.1 Full Shift Spaces and Bernoulli Shifts 89
6.2 Markov shifts 95
6.2.1 Subshifts of Finite Type 98
6.3 Markov Shifts in Higher Dimensions 99
6.4 Noninvertible Shifts 102
6.4.1 Index Function 102

7 Perron–Frobenius Theorem and Some Applications 107
7.1 Preliminary Background 108
7.2 Spectrum and the Perron–Frobenius Theorem 111
7.2.1 Application to Markov Shift Dynamics 114
7.3 An Application to Google's PageRank 116
7.4 An Application to Virus Dynamics 118
7.4.1 States of the Markov Process 118

8 Invariant Measures 125
8.1 Measures for Continuous Maps 125
8.2 Induced Transformations 131
8.3 Existence of Absolutely Continuous Invariant Probability Measures 135
8.3.1 Weakly Wandering Sets for Invertible Maps 139
8.3.2 Proof of the Hajian–Kakutani Weakly Wandering Theorem 142
8.4 Halmos–Hopf–von Neumann Classification 144

9 No Equivalent Invariant Measures: Type III Maps 151
9.1 Ratio Sets 152
9.2 Odometers of Type II and Type III 157
9.2.1 Krieger Flows 161
9.2.2 Type III_0 Dynamical Systems 163
9.3 Other Examples 164
9.3.1 Noninvertible Maps 165

10 Dynamics of Automorphisms of the Torus and Other Groups 167
10.1 An Illustrative Example 167
10.2 Dynamical and Ergodic Properties of Toral Automorphisms 171

10.3 Group Endomorphisms and Automorphisms on $\mathbb{T}^n$ 173
10.3.1 Ergodicity and Mixing of Toral Endomorphisms 177
10.4 Compact Abelian Group Rotation Dynamics 180

11 An Introduction to Entropy 185
11.1 Topological Entropy 188
11.1.1 Defining and Calculating Topological Entropy 188
11.1.2 Hyperbolic Toral Endomorphisms 192
11.1.3 Topological Entropy of Subshifts 194
11.2 Measure Theoretic Entropy 196
11.2.1 Preliminaries for Measure Theoretic Entropy 196
11.2.2 The Definition of $h_\mu(f)$ 197
11.2.3 Computing $h_\mu(f)$ 199
11.2.4 An Information Theory Derivation of $H(P)$ 203
11.3 Variational Principle 204
11.4 An Application of Entropy to the Papillomavirus Genome 205
11.4.1 Algorithm 206

12 Complex Dynamics 209
12.1 Background and Notation 211
12.1.1 Some Dynamical Properties of Iterated Functions 214
12.2 Möbius Transformations and Conformal Conjugacy 216
12.2.1 The Dynamics of Möbius Transformations 216
12.3 Julia Sets 217
12.3.1 First Properties of $J(R)$ 220
12.3.2 Exceptional and Completely Invariant Sets 222
12.3.3 Dynamics on Julia Sets 223
12.3.4 Classification of the Fatou Cycles 226
12.4 Ergodic Properties of Some Rational Maps 228
12.4.1 Ergodicity of Non-Critical Postcritically Finite Maps 233

13 Maximal Entropy Measures on Julia Sets and a Computer Algorithm 237
13.1 The Random Inverse Iteration Algorithm 238
13.2 Statement of the Results 239
13.3 Markov Processes for Rational Maps 243
13.3.1 Proof of Theorem 13.2 245
13.4 Proof That the Algorithm Works 246
13.5 Ergodic Properties of the Mañé–Lyubich Measure 247
13.6 Fine Structure of the Mañé–Lyubich Measure 248

14 Cellular Automata 255
14.1 Definition and Basic Properties 256
14.1.1 One-Dimensional CAs 256
14.1.2 Notation for Binary CAs with Radius 1 259
14.1.3 Topological Dynamical Properties of CA F_{90} 260
14.1.4 Measures for CAs 262

14.2 Equicontinuity Properties of CA 264
14.3 Higher Dimensional CAs 267
14.3.1 Conway's Game of Life 269
14.4 Stochastic Cellular Automata 270
14.5 Applications to Virus Dynamics 273

A Measures on Topological Spaces 279
A.1 Lebesgue Measure on $\mathbb{R}$ 279
A.1.1 Properties of m 280
A.1.2 A Non-measurable Set 283
A.2 Sets of Lebesgue Measure Zero 284
A.2.1 Examples of Null Sets 284
A.2.2 A Historical Note on Lebesgue Measure 285
A.3 The Definition of a Measure Space 286
A.4 Measures and Topology in Metric Spaces 287
A.4.1 Approximation and Extension Properties 290
A.4.2 The Space of Borel Probability Measures on X 291
A.4.3 Hausdorff Measures and Dimension 292
A.4.4 Some Useful Tools 293
A.5 Examples of Metric Spaces with Borel Measures 295
A.5.1 One-Dimensional Spaces 295
A.5.2 Discrete Measure Spaces 297
A.5.3 Product Spaces 297
A.5.4 Other Spaces of Interest 298

B Integration and Hilbert Spaces 303
B.1 Integration 303
B.1.1 Conventions About Values at ∞ and Measure 0 Sets 306
B.1.2 L^p Spaces 306
B.2 Hilbert Spaces 308
B.2.1 Orthonormal Sets and Bases 309
B.2.2 Orthogonal Projection in a Hilbert Space 309
B.3 Von Neumann Factors from Ergodic Dynamical Systems 312

C Connections to Probability Theory 315
C.1 Vocabulary and Notation of Probability Theory 315
C.2 The Borel-Cantelli Lemma 319
C.3 Weak and Strong Laws of Large Numbers 321

References 325

Index 333